# Osculating Circle with Microscopes Within Microscopes

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**Abstract.** Classically, an osculating circle at a point of a planar curve is introduced technically, often with formula giving its radius and the coordinates of its center. In this note, we propose a new and intuitive definition of this concept : among all the circles which have, on the considered point, the same tangent as the studied curve and thus seem equal to the curve through a microscope, the osculating circle is this that seems equal to the curve through a microscope within microscope.

# Osculating circle with microscopes within microscopes

For a long time, curves have been very important in the construction of mathematics, especially because they describe the motion of a geometrical point ; they were fundamental for the creation of the mathematical concept of function and the development of *differential and integral calculus*. The greatest mathematicians tried to characterize how a curve differs from a straight line and therefore they introduced the concept of *curvature* ; they constructed a circle which has the same tangent and the same curvature as the considered curve ; generally, such a circle exists and is unique : it is named, in latin and by Leibniz (1646 - 1716) himself, the *circulum osculans* (what has litteraly for traduction in English the "kissing circle") and is now known by the current name *osculating circle* : among all the tangent circles at the considered point, it is the circle that approaches the given curve most tightly.

## 1 Classical presentations

Firstly, we remind some classical definitions and results about the concept of osculating circle.

We consider a regular planar curve $\gamma\ : ]a,b[\mapsto \mathbb{R}^2$, with $\gamma(t) = (x(t), y(t))$.

In order to measure the failure of $\gamma$ to be a straight line, we introduce the (oriented) *curvature* of $\gamma$ which is denoted by $\kappa_2[\gamma](t)$ and is defined as follows ([4], p. 14)

$$\kappa_2[\gamma](t) = \frac{\gamma''(t).J\gamma'(t)}{||\gamma'(t)||^3}$$

where $J$ denoted the rotation by $\frac{\pi}{2}$ in a counterclockwise direction, i.e. the linear map $J : \mathbb{R}^2 \mapsto \mathbb{R}^2$ given by $J(p_1, p_2) = (-p_2, p_1)$.

The idea of curvature appeared implicitely in the geometrical work of the greek Apollonius (262 - 180 BC), but was first formally introduced by I. Newton (1642 - 1727) who had found this current formula

$$\kappa_2[\gamma](t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{\left[(x'(t))^2 + (y'(t))^2\right]^{\frac{3}{2}}}$$

In corollary, if $\gamma$ is the graph of a real function $f$, then

$$\kappa_2[\gamma](t) = \frac{f''(x)}{\left[1 + (f'(x))^2\right]^{\frac{3}{2}}}$$

The concept of curvature leads to the "best" circle that approximates the curve at a considered point. This circle, named *osculating circle*, is defined as follows at a point $\gamma(t)$ such that $a < t < b$ and $\kappa_2[\gamma](t) \neq 0$ ([4], p. 111) : it is the circle which has the radius $R(t)$ and the center $\Gamma(t)$ given by

$$R(t) = \frac{1}{|\kappa_2[\gamma](t)|} \quad \text{and } \Gamma(t) = \gamma(t) + R(t)\frac{J\gamma'(t)}{||\gamma'(t)||}$$

It is thus the circle for which the center lies on the normal line (i.e. on the perpendicular to the tangent) at a distance equal to the inverse, in absolute value, of the (oriented) curvature.

It is also possible to prove that the osculating circle of $\gamma$, with $a < t_0 < b$ and $\kappa_2[\gamma](t_0) \neq 0$, is the unique circle which has at least order 2 contact with $\gamma$ at $\gamma(t_0)$ ([4] p. 228). The concept of contact can be defined as follows ([4] p. 225). Let $F : \mathbb{R}^2 \mapsto \mathbb{R}$ be a differentiable function ; we say that the curve $\gamma$ and the implicitely curve $F^{-1}(0)$ have contact of order $n$ at $\gamma(t_0)$ when we have

$$(F \circ \gamma)(t_0) = (F \circ \gamma)'(t_0) = \ldots = (F \circ \gamma)^{(n)}(t_0) = 0$$

$$\text{but } (F \circ \gamma)^{(n+1)}(t_0) \neq 0$$

## 2 A new presentation

Now, we present an original definition of this concept : the starting point of our method is the use of powerful "microscopes", which give us the opportunity to visualize infinitesimal details of a curve in the neighbourhood of a point on this curve and to "see" an osculating circle. In other words, we make the most of the fact that the osculating circle has contact of order 2 at the curve, and thus this circle and this curve seem to be equal through a "microscope within a microscope".

Before that, we present in details the microscopes and the microscopes within microscopes, and then describe how these tools can be applied in order to determine an osculating circle. Moreover, we work with hyperreal numbers in the context of nonstandard analysis ; we first recall some fundamental and useful points about this modern theory.

Although Leibniz and Newton, for instance, already worked with "infinitesimals", these numbers were only rigorously introduced in 1961 by A. Robinson [8], who developed the "nonstandard analysis". Here below, we give the essential elements of this theory and work with one of the didactical and simple presentation by Keisler ([5]) : we shall essentially adopt his definitions and notations.

First, we recall that the hyperreal numbers extend the real ones with the same algebraic rules ; technically, the set $^*\mathbb{R}$ of the hyperreal numbers is a non-archimedean ordered field in which the real line $\mathbb{R}$ is embedded. Moreover, $^*\mathbb{R}$ contains at least one (in fact infinitely many) *infinitesimal*, i.e. a number $\varepsilon$ such that its absolute value is less than every positive real number (or, formally, $0 < |\varepsilon| < r, \forall r \in \mathbb{R}_+ = \{x \in \mathbb{R} \ : x > 0\}$); its reciproqual $H = \frac{1}{\varepsilon}$ is *infinitely huge* or, simply, *infinite*, i.e. it is a number such that its absolute value is greater than every real number (or, formally, $|H| > r, \forall r \in \mathbb{R}$) ; clearly, non-zero infinitesimals and infinite numbers are not real. A hyperreal number $x$ which is not infinite is of course said to be *finite* ; in this case, there exists one and only one real number $r$ which is *infinitely close* to $x$, i.e. such that the difference $x - r$ is an infinitesimal : $r$ is called the *standard part* of $x$ and is denoted by $r = \text{st}(x)$ ; formally, st is a ring homomorphism from the set of finite hyperreal numbers to $\mathbb{R}$ and its kernel is the set of infinitesimals. Moreover, every function of one or several real variables has a *natural extension* into hyperreal numbers, with the same definition and the same properties as these of the original one ; in this note, we adopt the same notation for a real function and for its natural extension.

The concept of (virtual) microscope is rather well-known (see, for example, [1], [2], [3]). For a point $P(a, b)$ in the hyperreal plane $^*\mathbb{R}^2$ and a positive infinite hyperreal number $H$, a $H$-microscope *pointed* on $P$ and with $H$ as *power* magnifies the distances from $P$ by a factor $H$ : it is denoted by $\mathcal{M}_H^P$ ; technically, it works as a map, denoted by $\mu$, defined on $^*\mathbb{R}^2$ as follows

$$\mu : (x, y) \mapsto \mu(x, y) = (H(x - a), H(y - b))$$

For a real function $f$ of two real variables $x$ and $y$, we first consider, in the classical euclidean plane $\mathbb{R}^2$, the curve $\mathcal{C}$ defined by

$$f(x,y) = 0$$

We consider a point $P(r,s)$ belonging to $\mathcal{C}$ and we suppose that $f$ is of class $C^p$, for $p$ sufficiently large, in a neighbourhood of $P$. For more simplicity, we denote by $d_1$, $d_2$, $d_{11}$, $d_{12}$ and $d_{22}$ the corresponding partial derivatives of $f$ at $P$, i.e. $d_1 = f_x(r,s)$, $d_2 = f_y(r,s)$, $d_{11} = f_{xx}(r,s)$, $d_{12} = f_{xy}(r,s)$ and $d_{22} = f_{yy}(r,s)$. Moreover, we assume that $d_2 \neq 0$.

When looking at $\mathcal{C}$ through the microscope $\mathcal{M}_H^P$, where $H$ is an arbitrary positive infinite hyperreal number, we indeed apply the map $\mu_1 : (x,y) \mapsto (X,Y))$ with

$$X = H\ \ (x-r) \Leftrightarrow x = r + \frac{X}{H} \text{ and } Y = H\ \ (y-s) \Leftrightarrow y = s + \frac{Y}{H}$$

So, we see the curve characterized, with the new coordinates $X$ and $Y$, by the equation

$$f\left(r + \frac{X}{H}, s + \frac{Y}{H}\right) = 0$$

Taylor's Formula for $f$ at $P$ and easy computations lead to

$$d_1\ \frac{X}{H} + d_2\ \frac{Y}{H} + \frac{\varepsilon}{H} = 0$$

where $\varepsilon$ denotes an infinitesimal number. After a multiplication by $H$, we take the standard parts of the two members of this equality in order to obtain the real image in the ocular of the microscope :

$$d_1\ \mathrm{st}(X) + d_2\ \mathrm{st}(Y) = 0 \Longleftrightarrow \mathrm{st}(Y) = -\frac{d_1}{d_2}\ \mathrm{st}(X)$$

It is well-known that the coefficient $m = -\frac{d_1}{d_2}$ is the slope of the tangent line $\mathcal{T}$ to $\mathcal{C}$ at $P$ ; $\mathcal{T}$ is thus defined by this equation

$$y - s = m(x-r)$$

In order to distinguish between the curve $\mathcal{C}$ and its tangent $\mathcal{T}$, we use a stronger microscope, for example with a power $H^2$ ; so, we use what is named a "microscope within microscope" ([2], [3]). We must point it on a point which differs from $P$, otherwise we see again $\mathcal{C}$ and $\mathcal{T}$ as equal ; here, we do not choose a point on the curve, as in [3], but we consider the point $P_1\left(r + \frac{1}{H}, s + \frac{m}{H}\right)$ which is infinitely close to $P$ and belongs to $\mathcal{T}$ : so, we work with the microscope $\mathcal{M}_{H^2}^{P_1}$ ; we can also follow the same reasoning with the point $P_2\left(r - \frac{1}{H}, s - \frac{m}{H}\right)$ and the microscope $\mathcal{M}_{H^2}^{P_2}$, but we only develop here the first case. Thus, we use a new map $\mu_2$ from ${}^*\mathbb{R}^2$ to ${}^*\mathbb{R}^2$ defined by

$$\mu_2 : (x,y) \mapsto (\mathcal{X}, \mathcal{Y})$$

with

$$\mathcal{X} = H^2\left(x - r - \frac{1}{H}\right) \text{ and } \mathcal{Y} = H^2\left(y - s - \frac{m}{H}\right)$$

When we apply this microscope $\mathcal{M}_{H^2}^{P_1}$, we work with new coordinates $\mathcal{X}$ and $\mathcal{Y}$. The image of the tangent $\mathcal{T}$ still is the line through the origin of the plane and the slope of which is $m$ ; for the curve $\mathcal{C}$, we get

$$\begin{array}{rcl} 0 & = & f\left(r+\frac{1}{H}+\frac{\mathcal{X}}{H^2}, s+\frac{m}{H}+\frac{\mathcal{Y}}{H^2}\right) \\ & = & f\left(r,s\right)+d_1\left(\frac{1}{H}+\frac{\mathcal{X}}{H^2}\right)+d_2\left(\frac{m}{H}+\frac{\mathcal{Y}}{H^2}\right)+ \\ & & \frac{1}{2}\left[d_{11}\ \left(\frac{1}{H}+\frac{\mathcal{X}}{H^2}\right)^2+2d_{12}\left(\frac{1}{H}+\frac{\mathcal{X}}{H^2}\right)\ \left(\frac{m}{H}+\frac{\mathcal{Y}}{H^2}\right)+d_{22}\ \left(\frac{m}{H}+\frac{\mathcal{Y}}{H^2}\right)^2\right]+\frac{\varepsilon_1}{H^2} \end{array}$$

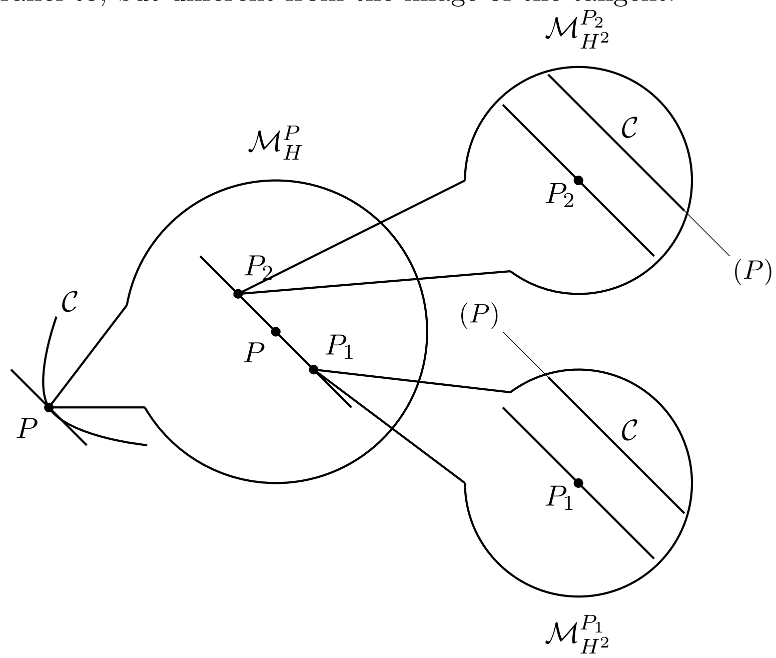
where $\varepsilon_1$ still denotes an infinitesimal. We easily have thanks to easy algebraic computations and to the rules of the standard part

$$d_1\ \mathrm{st}(\mathcal{X})+d_2\ \mathrm{st}(\mathcal{Y})+\frac{1}{2}\left[d_{11}+2\ m\ d_{12}+m^2\ d_{22}\right]=0$$

or equivalently

$$\mathrm{st}(\mathcal{Y})=m\ \mathrm{st}(\mathcal{X})-\frac{1}{2d_2}\left[d_{11}+2\ m\ d_{12}+m^2\ d_{22}\right]$$

It is a line parallel to, but different from the image of the tangent.




We now consider an arbitrary circle which has at the point $P$ the straight line $\mathcal{T}$ as tangent ; for simplicity, we call *tangent circle* such a circle. Its center lies on the inner normal line defined by this equation

$$y-s=\frac{-1}{m}\left(x-r\right)\quad \text{if } m\neq 0$$

on the vertical line $x=r$ if $m=0$. Note that, in the following, we only consider the general case $m\neq 0$, because the particular case $m=0$ is obvious. If its radius is the number $R$ and its center is the point $(a,b)$ (with $b\neq s$), we have

$$R^2=\left(a-r\right)^2+\left(b-s\right)^2\ \text{ and } b-s=\frac{-1}{m}\left(a-r\right)$$

Moreover, its image through $\mathcal{M}_{H^2}^{P_1}$ , i.e. by the map $\mu_2$ introduced here above, is given by

$$\left(r+\frac{1}{H}+\frac{\mathcal{X}}{H^2}-a\right)^2+\left(s+\frac{m}{H}+\frac{\mathcal{Y}}{H^2}-b\right)^2=\left(a-r\right)^2+\left(b-s\right)^2$$

Elementary algebraic computations and the rules of the standard parts easily lead to

$$2\,(r-a)\,\mathrm{st}(\mathcal{X})+2\,(s-b)\,\mathrm{st}(\mathcal{Y})+1+m^2=0 \Longleftrightarrow \mathrm{st}(\mathcal{Y})=-\frac{r-a}{s-b}\mathrm{st}(\mathcal{X})-\frac{1+m^2}{2\,(s-b)}$$

It also is equivalent to

$$\mathrm{st}(\mathcal{Y})=m\ \mathrm{st}(\mathcal{X})-\frac{1+m^2}{2\,(s-b)}$$

We here define the osculating circle at $P$ on the curve $\mathcal{C}$ as follows : it is the unique circle among all these tangent circles which has the same image by the map $\mu_2$ as $\mathcal{C}$ ; in other words, when we look at $\mathcal{C}$ and at a tangent circle through $\mathcal{M}_{H^2}^{P_1}$ (and this is the same with $\mathcal{M}_{H^2}^{P_2}$), we must see an unique straight line.

By the comparison of two preceding formulas, we have

$$-\frac{1}{2d_2}\left[d_{11}+2\ m\ d_{12}+m^2\ d_{22}\right]=-\frac{1+m^2}{2\,(s-b)}$$

So, we find the center of the osculating circle : its coordinates are given by these equalities

$$b-s=-\frac{\left(1+m^2\right)d_2}{d_{11}+2\ m\ d_{12}+m^2\ d_{22}} \text{ and } a-r=-m\,(b-s)=m\frac{\left(1+m^2\right)d_2}{d_{11}+2\ m\ d_{12}+m^2 d_{22}}$$

The square of the radius $R$ of the osculating circle is equal to $\left(1+m^2\right)(s-b)^2$ ; therefore, we have

$$R=\left(1+m^2\right)^{\frac{3}{2}}\left|\frac{d_2}{d_{11}+2\ m\ d_{12}+m^2\ d_{22}}\right|=\frac{\left(d_1^2+d_2^2\right)^{\frac{3}{2}}}{|d_2^2 d_{11}-2d_1 d_2 d_{12}+d_1^2 d_{22}|}$$

In particular, when the curve $\mathcal{C}$ is the graph of a function $f$, i.e. when we have $y-f(x)=0$, this last formula gives

$$R=\frac{\left[1+(f'(x))^2\right]^{\frac{3}{2}}}{f''(x)}$$

which is equivalent to the formula given previously.

Here is an elementary and illuminating example.

We consider the point $P\,(0,-1)$ on the ellipse $\mathcal{E}$ defined by

$$x^2+4y^2=4$$

The tangent $\mathcal{T}$ to $\mathcal{E}$ on $P$ is the horizontal straight line $y=-1$. With the notations used in the paper, we get

$$f\,(x,y)=x^2+4y^2-4\ ,\ d_1=d_{12}=m=0\ ,\ d_2=-8\ ,\ d_{11}=2\ ,\ d_{22}=8\ ,\ R=\frac{|-8|}{2}=4$$

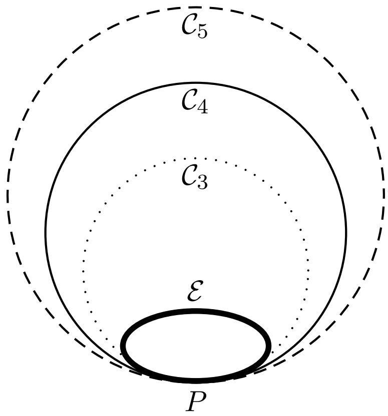
By applying a microscope within microscope, i.e. $\mathcal{M}_{H^2}^{P_1}$ with $P_1\left(\frac{1}{H},-1\right)$, we see the image of the ellipse $\mathcal{E}$ as the horizontal line with the vertical coordinate equal to $\frac{1}{8}$.

Besides, the circles which are tangent to $\mathcal{E}$ at $P$ have their center on the vertical axis of the plane. More precisely, denote by $\mathcal{C}_a$ the circle of which the center is $(0, -1+a)$ and the radius is $a$ (with $a > 0$) ; its equation can be given by

$$x^2 + (y+1-a)^2 = a^2 \Longleftrightarrow x^2 + y^2 + 2\,(1-a)\,y + 1 - 2a = 0$$

The image of $\mathcal{C}_a$ in the ocular of $\mathcal{M}_{H^2}^{P_1}$ is the horizontal line with the vertical coordinate equal to $\frac{1}{2a}$. Thus, the osculating circle of $\mathcal{E}$ at $P$ is $\mathcal{C}_4$, since $2a = 8$.

The real situation can be seen on this figure : it is clear that, in the neigbourhood of $P$, $\mathcal{C}_4$ is indeed a better approximation of $\mathcal{E}$ than $\mathcal{C}_3$ or $\mathcal{C}_5$.




## Conclusion

In conclusion, we observe that microscopes and microscopes within microscopes are interesting "epistemic mediators" ([6], [7]) : they allow us to "discover through doing" several and important concepts in the study of curves, like the concepts of tangent, the convexity or concavity, inflexion points ([1]). Here, we have shown that these tools can also be used in order to find the curvature and the osculating circle for a curve.

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